

An approximately truthful-in-expectation mechanism for combinatorial auctions using value queries

Shaddin Dughmi* Tim Roughgarden† Jan Vondrák‡ Qiqi Yan§

September 7, 2011

1 Introduction

This manuscript presents an alternative implementation of the truthful-in-expectation (TIE) mechanism of Dughmi, Roughgarden and Yan [2] for combinatorial auctions. Recall that in a *combinatorial auction*, m goods get allocated to n bidders. Each bidder i has a private valuation v_i that describes its value $v_i(S)$ for each subset S of goods. The social welfare of an allocation is the sum of the bidders' values for the goods received.

The mechanism of [2] is presented in a “lottery-value” oracle model, where each bidder can be queried about his valuation by means of the following query: given a vector of probabilities over items $\mathbf{x} \in [0, 1]^m$, what is the expected value $\mathbf{E}[v_i(\hat{\mathbf{x}})]$, where $\hat{\mathbf{x}}$ is obtained by independently rounding each coordinate of \mathbf{x} to 0 or 1 with probability x_i . Such queries can be answered efficiently for certain valuation functions (in particular *coverage* functions), and this oracle model is a convenient framework for the presentation of the mechanism of [2]. On the other hand, lottery-value queries are #P-hard to answer for the class of matroid rank functions (see Section 4), and hence one can ask how realistic this model is in general. The purpose of this manuscript is to show that the model is “approximately realistic” in the sense that the mechanism of [2] can be implemented in the (weaker) value oracle model at the cost of relaxing the concept of truthfulness in expectation to *approximate truthfulness in expectation*. (Here, we mean approximation within an arbitrarily small error, in the sense of an FPTAS.) In particular, we show that the maximal in distributional range (MIDR) allocation rule of [2] can be implemented as an approximately MIDR allocation rule in the value oracle model, and then we present a blackbox transformation from approximately MIDR allocation rules to approximately TIE mechanisms.

First, let us define the approximate variants of MIDR and truthfulness in expectation. The exact variants are obtained by setting $\epsilon = 0$.

Definition 1.1. *An allocation rule is $(1 - \epsilon)$ -approximately maximal-in-distributional range (or $(1 - \epsilon)$ -MIDR) if there is a range of distributions over outcomes \mathcal{R} such that, for every input, the mechanism returns an outcome that is sampled from a distribution $D^* \in \mathcal{R}$ that $(1 - \epsilon)$ -approximately maximizes the expected social welfare $\mathbf{E}_{\omega \sim D}[\sum_i v_i(\omega)]$ over all distributions $D \in \mathcal{R}$.*

We define approximate truthfulness in expectation in terms of *relative utility error*. This means that truth-telling costs a bidder at most an ϵ fraction of his maximum-possible utility.

Definition 1.2. *A mechanism with allocation and payment rules A and p is $(1 - \epsilon)$ -approximately truthful-in-expectation (or $(1 - \epsilon)$ -TIE) if, for every bidder i , (true) valuation function v_i , (reported) valuation*

*Microsoft Research, Redmond, WA

†Stanford University, Stanford, CA

‡IBM Almaden Research Center, San Jose, CA

§Stanford University, Stanford, CA

function v'_i , and (reported) valuation functions v_{-i} of the other bidders,

$$\mathbf{E}[v_i(A(v_i, v_{-i})) - p_i(v_i, v_{-i})] \geq (1 - \epsilon) \mathbf{E}[v_i(A(v'_i, v_{-i})) - p_i(v'_i, v_{-i})]. \quad (1)$$

The expectation in (1) is over the coin flips of the mechanism.

The class of valuations of interest here is the following (as in [2]).

Definition 1.3. A function $v : 2^{[m]} \rightarrow \mathbb{R}_+$ is a weighted matroid rank sum if there are matroids $\mathcal{M}_1, \dots, \mathcal{M}_k$ and weights $\alpha_1, \dots, \alpha_k \geq 0$ such that

$$v(S) = \sum_{i=1}^k \alpha_i r_{\mathcal{M}_i}(S),$$

where $r_{\mathcal{M}_i}$ is the rank function of matroid \mathcal{M}_i .

This definition also captures positive combinations of *weighted rank functions*, as the cones generated by weighted and unweighted rank functions of matroids coincide.

The allocation rule of [2] is maximal-in-distributional-range, provides a $(1 - 1/e)$ -approximation to the social welfare, and the corresponding TIE mechanism can be implemented in expected polynomial time provided the bidders' valuations are weighted matroid rank sums and support lottery-value queries. Our goal here is to prove the following.

Theorem 1.4. For every $\epsilon = 1/\text{poly}(m, n)$, there is a $(1 - \epsilon)$ -TIE mechanism that achieves a $(1 - 1/e - \epsilon)$ -approximation to the social welfare in combinatorial auctions with weighted-matroid-rank-sum valuations that runs in polynomial time in the value oracle model.

We prove this claim in two steps. First, we prove the following.

Theorem 1.5. For every $\epsilon = 1/\text{poly}(m, n)$, there is a $(1 - \epsilon)$ -MIDR allocation rule that achieves a $(1 - 1/e - \epsilon)$ -approximation to the social welfare in combinatorial auctions with weighted-matroid-rank-sum bidders and runs in polynomial time in the value oracle model.

Then, we prove the following general reduction.

Theorem 1.6. For every $(1 - \epsilon)$ -MIDR allocation rule that is a c -approximation to the social welfare in combinatorial auctions with bidders' valuations restricted to a set \mathcal{C} , there is a $(1 - \epsilon')$ -TIE mechanism that is a $(c - 1/\text{poly}(m, n))$ -approximation to the social welfare in combinatorial auctions with bidders' valuations in \mathcal{C} , where $\epsilon' = \epsilon \cdot \text{poly}(m, n)$.

Therefore, by selecting a sufficiently (polynomially) small ϵ in Theorem 1.5, we can achieve an arbitrarily (polynomially) small error ϵ in Theorem 1.4.

2 An Approximately MIDR Allocation Rule

In this section, we prove Theorem 1.5. We use a variant of the mechanism of [2]. Instead of sophisticated convex optimization techniques, which seem necessary to find the exact optimum over the range, we use a simple local search that guarantees that we get arbitrarily close to the optimum. We begin with some definitions.

Definition 2.1. For a combinatorial auction with m items and n bidders with valuations $v_i : 2^{[m]} \rightarrow \mathbb{R}_+$, the aggregate valuation function $f : 2^{[n] \times [m]} \rightarrow \mathbb{R}_+$ is

$$f(S) = \sum_{i=1}^n v_i(\pi_i(S)),$$

where $\pi_i(S) = \{j : (i, j) \in S\}$. We define $F : [0, 1]^{[n] \times [m]} \rightarrow \mathbb{R}_+$ to be the multilinear extension of f (see also [5]), and P to be the polytope of fractional allocations:

$$P = \left\{ \mathbf{x} \in [0, 1]^{[n] \times [m]} : \forall j : \sum_{i=1}^n x_{ij} \leq 1 \right\}.$$

The (integral) welfare maximization problem turns out to be equivalent to $\max\{F(\mathbf{x}) : \mathbf{x} \in P\}$. This problem cannot be solved optimally, even for very special classes of valuation functions. In lieu of F , the authors of [2] use the modified objective function

$$F^{exp}(x_{11}, x_{12}, \dots, x_{nm}) = F(1 - e^{-x_{11}}, 1 - e^{-x_{12}}, \dots, e^{-x_{nm}}).$$

Interestingly, the function F^{exp} turns out to be *concave* for a subclass of submodular functions, including weighted matroid rank sums (see [2] for a proof). This means that we can solve the problem $\max\{F^{exp}(\mathbf{x}) : \mathbf{x} \in P\}$, which means in effect optimizing over a certain range of product distributions. Also, the optimum of this problem is at least $(1 - 1/e)$ times the optimal social welfare. Supplementing this MIDR allocation rule with suitable payments yields a $(1 - 1/e)$ -approximate, TIE mechanism [2].

Here, we propose the following simple algorithm that solves the problem $\max\{F^{exp}(\mathbf{x}) : \mathbf{x} \in P\}$ near-optimally (in the sense of an FPTAS).

Local Search Allocation Rule.

- Initialize $\mathbf{x} := 0$. Let M be the maximum value of any singleton.
- Let \mathbf{g} be an estimate of the gradient $\nabla F^{exp}(\mathbf{x})$, within additive error δM in each coordinate, where $\delta = \frac{\epsilon}{8m^2n^2}$. As long as there is a point $\mathbf{y} \in P$ such that

$$(\mathbf{y} - \mathbf{x}) \cdot \mathbf{g} > \frac{1}{2}\epsilon M,$$

update $\mathbf{x} := \mathbf{x} + \delta(\mathbf{y} - \mathbf{x})$.

- Return an allocation randomly sampled from the distribution \mathbf{x} .

The required estimates of $\nabla F^{exp}(\mathbf{x})$ can be obtained in polynomial time by random sampling, with high probability. (By *high probability*, we mean $1 - e^{-poly(m, n)}$ in this manuscript. The coordinates of $\nabla F^{exp}(\mathbf{x})$ are always in the interval $[0, M]$ — by submodularity — and so this follows from standard Chernoff bounds [1].) Linear programming can be used to efficiently find a suitable point \mathbf{y} , or certify that no such point exists.

2.1 The analysis

We claim that this allocation rule runs in polynomial time and solves the problem $\max\{F^{exp}(\mathbf{x}) : \mathbf{x} \in P\}$ up to a $(1 - \epsilon)$ factor with high probability, thus proving Theorem 1.5. In the following, we assume that the estimate of $\nabla F^{exp}(\mathbf{x})$ obtained in each step is accurate within additive error δM , which is possible to achieve with high probability over the run of the algorithm, via a polynomial number of samples.

We proceed via a series of claims.

Lemma 2.2. *If the algorithm terminates, then with high probability*

$$F^{exp}(\mathbf{x}) \geq (1 - \epsilon) \max\{F^{exp}(\mathbf{x}) : \mathbf{x} \in P\}.$$

Proof. Let \mathbf{y} be an optimal solution of $\max\{F^{exp}(\mathbf{x}) : \mathbf{x} \in P\}$. When the algorithm terminates at \mathbf{x} , we have $(\mathbf{y} - \mathbf{x}) \cdot \nabla F^{exp}(\mathbf{x}) < \epsilon M$ (even accounting for the errors in our estimate of $\nabla F^{exp}(\mathbf{x})$). By the concavity of F^{exp} ,

$$OPT - F^{exp}(\mathbf{x}) = F^{exp}(\mathbf{y}) - F^{exp}(\mathbf{x}) \leq (\mathbf{y} - \mathbf{x}) \cdot \nabla F^{exp}(\mathbf{x}) \leq \epsilon M \leq \epsilon OPT.$$

□

Lemma 2.3. *In each iteration, with high probability, the value of $F^{exp}(\mathbf{x})$ increases by at least $\frac{\epsilon^2}{64m^2n^2}M$.*

Proof. If the algorithm continues, we can assume that $(\mathbf{y} - \mathbf{x}) \cdot \nabla F^{exp}(\mathbf{x}) > \frac{1}{4}\epsilon M$ (considering that the estimate of $\nabla F^{exp}(\mathbf{x})$ could be off by $\delta M = \frac{\epsilon M}{8m^2n^2}$ in each coordinate). We also have bounds on how much the gradient can change when \mathbf{x} moves by a certain amount. Specifically, for $(i, j) \neq (i', j')$,

$$\left| \frac{\partial^2 F^{exp}}{\partial x_{ij} \partial x_{i'j'}} \right| = \left| e^{-x_{ij} - x_{i'j'}} \frac{\partial^2 F}{\partial x_{ij} \partial x_{i'j'}} \right| \leq \left| \frac{\partial^2 F}{\partial x_{ij} \partial x_{i'j'}} \right| \leq M$$

and similarly

$$\left| \frac{\partial^2 F^{exp}}{\partial x_{ij}^2} \right| = \left| e^{-x_{ij}} \frac{\partial^2 F}{\partial x_{ij}^2} \right| \leq \left| \frac{\partial^2 F}{\partial x_{ij}^2} \right| \leq M,$$

by known properties of the multilinear extension [5, 6]. This implies that for any \mathbf{x}' such that $\|\mathbf{x}' - \mathbf{x}\|_\infty \leq \delta$,

$$\left. \frac{\partial F^{exp}}{\partial x_{ij}} \right|_{\mathbf{x}'} \geq \left. \frac{\partial F^{exp}}{\partial x_{ij}} \right|_{\mathbf{x}} - \sum_{i,j} |x'_{ij} - x_{ij}| \max \left| \frac{\partial^2 F^{exp}}{\partial x_{ij} \partial x_{i'j'}} \right| \geq \left. \frac{\partial F^{exp}}{\partial x_{ij}} \right|_{\mathbf{x}} - \delta mnM.$$

Consequently,

$$\begin{aligned} F^{exp}(\mathbf{x} + \delta(\mathbf{y} - \mathbf{x})) &\geq F^{exp}(\mathbf{x}) + \delta(\mathbf{y} - \mathbf{x}) \cdot (\nabla F^{exp}(\mathbf{x}) - \delta mnM \mathbf{1}) \\ &\geq F^{exp}(\mathbf{x}) + \delta(\mathbf{y} - \mathbf{x}) \cdot \nabla F^{exp}(\mathbf{x}) - \delta^2 m^2 n^2 M \\ &\geq F^{exp}(\mathbf{x}) + \delta \cdot \frac{1}{4}\epsilon M - \delta^2 m^2 n^2 M. \end{aligned}$$

Again using $\delta = \frac{\epsilon}{8m^2n^2}$, we obtain

$$F^{exp}(\mathbf{x} + \delta(\mathbf{y} - \mathbf{x})) \geq F^{exp}(\mathbf{x}) + \frac{\epsilon^2}{32m^2n^2}M - \frac{\epsilon^2}{64m^2n^2}M \geq F^{exp}(\mathbf{x}) + \frac{\epsilon^2}{64m^2n^2}M.$$

□

Lemma 2.4. *The number of iterations is with high probability at most $64m^3n^2/\epsilon^2$.*

Proof. By the previous lemma, with high probability the value of $F^{exp}(\mathbf{x})$ increases in each iteration by at least $\frac{\epsilon^2}{64m^2n^2}M$. After $64m^3n^2/\epsilon^2$ iterations, it will be at least mM . By the definition of M and submodularity of valuations, mM is an upper bound on the welfare of every feasible allocation, and hence also of the function F^{exp} . This completes the proof. □

This concludes the proof of Theorem 1.5. We remark that, building on the allocation rule in [3], a similar approach gives a $(1 - \epsilon)$ -MIDR and $(1 - 1/e - \epsilon)$ -approximate allocation rule for combinatorial public projects with weighted matroid rank sums that runs in polynomial time in the value oracle model.

3 From Approximately MIDR to Approximately TIE

In this section, we prove Theorem 1.6. We assume that we have a $(1 - \epsilon)$ -MIDR allocation rule \mathcal{M} providing a c -approximation for combinatorial auctions with valuations in a class \mathcal{C} . We assume in the following that $c \geq \frac{1}{n}$, where n is the number of bidders. (A $\frac{1}{n}$ -approximation is trivial to achieve by giving all of the items to a random bidder.) We also assume that $\epsilon = 1/\text{poly}(m, n)$. We want to convert the $(1 - \epsilon)$ -MIDR allocation rule into an $(1 - \epsilon')$ -TIE mechanism. Our approach is as follows. If $\epsilon = 0$, then the VCG payment scheme turns an MIDR mechanism into a TIE mechanism. The fact that our mechanism is only approximately MIDR means that the VCG payments might suffer from errors that are significant for certain bidders, especially if their utility is close to zero. Therefore, we modify the mechanism to ensure that bidders whose valuation is very low do not participate in the VCG scheme. In addition, we provide each bidder with the bundle of *all* items with some small probability, so that their expected utility is not extremely small. Our mechanism works as follows.

Mechanism \mathcal{M}' .

1. Let \mathcal{M} be a $(1 - \epsilon)$ -MIDR allocation rule.
2. Let V_i be the valuation that bidder i reports for the ground set of all items. Run \mathcal{M} to compute a distribution over allocations (S_1, \dots, S_n) and let O_i be an (unbiased) estimate of the expected value collected by bidder i , $\mathbf{E}[v_i(S_i)]$.¹ Let $O = \sum_{i=1}^n O_i$ be an estimate of the expected social welfare $\sum_{i=1}^n \mathbf{E}[v_i(S_i)]$. For each bidder i , run \mathcal{M} also on the same instance without bidder i , and denote by O'_{-i} an estimate of the expected social welfare of its outcome.
3. Call bidder i *relevant* if

$$V_i > \frac{1}{n^7} \sum_{j \neq i} V_j.$$

Call bidder i *active* if he is relevant and, in addition,

$$\left(1 - \frac{1}{n}\right)(O - O'_{-i}) + \frac{1}{2n^2}V_i > \frac{1}{n^4}O'_{-i}.$$

4. With probability $1 - 1/n$: allocate a set S_i from the distribution found by \mathcal{M} to each active bidder i , and charge the VCG-like price $p_i = O'_{-i} - \sum_{j \neq i} O_j$. Do not allocate or charge anything to inactive bidders.
5. Else, with probability $1/n^2$ for each bidder i : If active, allocate the ground set to i and charge $\frac{1}{n^2}O'_{-i}$. If inactive, allocate the ground set with probability $1/2$ and charge 0.

We emphasize that O and O'_{-i} are random variables that we obtain by running the (randomized) allocation rule \mathcal{M} . We denote the actual optima over the range, with respect to the reported valuations, by OPT and OPT'_{-i} . In expectation, we have $OPT \geq \mathbf{E}[O] \geq (1 - \epsilon)OPT$ and $OPT'_{-i} \geq \mathbf{E}[O'_{-i}] \geq (1 - \epsilon)OPT'_{-i}$; however with some probability, O could be significantly different from OPT (even larger, since it is a probabilistic estimate), and O'_{-i} could be significantly different from OPT'_{-i} .

In the following, we denote by v_i^* the actual valuation of bidder i , and by $V_i^* = v_i^*(M)$ the actual value of the ground set for bidder i . Let OPT_{+i} denote the optimum over the range with valuations v_j for $j \neq i$ and v_i^* for bidder i . (Note that $OPT_{+i} = OPT$ if $v_i^* = v_i$.) Let O_{+i} denote our estimate of OPT_{+i} (assuming bidder i reports the truth). We prove the following statements.

Lemma 3.1. *For every bidder such that $V_i^* \leq \frac{1}{n^7} \sum_{j \neq i} V_j$, his expected utility is maximized within an ϵ -fraction of his utility by reporting truthfully.*

Proof. Observe that as long as bidder i reports $V_i \leq \frac{1}{n^7} \sum_{j \neq i} V_j$, he is inactive and receives the same utility regardless of his bid. Therefore, his utility could change only if he reports $V_i > \frac{1}{n^7} \sum_{j \neq i} V_j$. In that case, he might be classified as active (depending on O and O'_{-i}). However, if that happens then he is charged at least $\frac{1}{n^2}O'_{-i}$ with probability $\frac{1}{n^2}$, i.e. $\frac{1}{n^4}O'_{-i}$ in expectation (conditioned on the value of O'_{-i}). Since the most value he can ever collect is V_i^* , he would (possibly) gain from being active only if $O'_{-i} < n^4 V_i^* \leq \frac{1}{n^3} \sum_{j \neq i} V_j$. Since $\mathbf{E}[O'_{-i}] \geq (1 - \epsilon)OPT'_{-i} > \frac{1}{n^2} \sum_{j \neq i} V_j$, it is very unlikely that O'_{-i} is less than $\frac{1}{n^3} \sum_{j \neq i} V_j$; this happens with exponentially small probability. Hence, by lying, bidder i could possibly gain only an exponentially small fraction of V_i^* in expectation, negligible with respect to his utility as an inactive player. \square

Lemma 3.2. *For every bidder i such that $V_i^* > \frac{1}{n^7} \sum_{j \neq i} V_j$, we have*

$$\mathbf{E}[|OPT_{+i} - O_{+i}|] \leq 3\epsilon n^7 V_i^*$$

and

$$\mathbf{E}[|OPT'_{-i} - O'_{-i}|] \leq 2\epsilon n^7 V_i^*.$$

¹By polynomially bounded sampling, we can assume that our estimate O_i is with high probability within $\mathbf{E}[v_i(S_i)] \pm V_i/\text{poly}(m, n)$, and the probability of deviation decays exponentially. Similarly for the estimates of O and O'_{-i} .

Proof. We have $V_i^* > \frac{1}{n^7} \sum_{j \neq i} V_j \geq \frac{1}{n^7} OPT'_{-i}$. We also know that $|OPT'_{-i} - \mathbf{E}[O'_{-i}]| \leq \epsilon OPT'_{-i} \leq \epsilon n^7 V_i^*$. The estimate O'_{-i} of the output of the mechanism for all bidders except i is concentrated around its expectation $\mathbf{E}[O'_{-i}]$, with variance $\frac{1}{\text{poly}(m,n)} \sum_{j \neq i} V_j \ll \epsilon n^7 V_i^*$, hence we can estimate

$$\mathbf{E}[|OPT'_{-i} - O'_{-i}|] \leq 2\epsilon n^7 V_i^*.$$

Similarly, $|OPT_{+i} - \mathbf{E}[O_{+i}]| \leq \epsilon OPT_{+i} \leq \epsilon(OPT_{-i} + V_i^*) \leq 2\epsilon n^7 V_i^*$, and O_{+i} is concentrated around its expectation, therefore $\mathbf{E}[|OPT_{+i} - O_{+i}|] \leq 3\epsilon n^7 V_i^*$. \square

Lemma 3.3. *Every bidder i such that $V_i^* > \frac{1}{n^7} \sum_{j \neq i} V_j$ maximizes his expected utility within a factor of $(1 - O(\epsilon n^9))$ by reporting his true valuation.*

Proof. Let us fix the valuations of all bidders except i . Let us assume for now that our estimate O_j is exactly equal to $\mathbf{E}[v_j(S_j)]$, $O = \sum_{j=1}^n O_j$ is equal to OPT (meaning that the MIDR mechanism optimizes exactly), and similarly O'_{-i} is equal to OPT'_{-i} . We will analyze this idealized mechanism first.

If bidder i ends up being active, his expected utility will be

$$\begin{aligned} U_{\text{active}} &= (1 - 1/n)(\mathbf{E}[v_i^*(S_i)] - (O'_{-i} - \sum_{j \neq i} O_j)) + \frac{1}{n^2} V_i^* - \frac{1}{n^4} OPT'_{-i} \\ &= (1 - 1/n)(\mathbf{E}[v_i^*(S_i)] + \sum_{j \neq i} \mathbf{E}[v_j(S_j)] - OPT'_i) + \frac{1}{n^2} V_i^* - \frac{1}{n^4} OPT'_{-i} \\ &\leq (1 - 1/n)(OPT_{+i} - OPT'_{-i}) + \frac{1}{n^2} V_i^* - \frac{1}{n^4} OPT'_{-i} = U_{\text{active}}^+. \end{aligned}$$

Here we used the fact that OPT_{+i} is the optimal value over the range with valuations v_i^* and $v_j, j \neq i$. This implies that the last quantity, U_{active}^+ , is the best possible utility bidder i could receive as an active bidder. In fact he will receive this utility if he reports truthfully and ends up being active.

If bidder i is inactive, then his expected utility will be

$$U_{\text{inactive}} = \frac{1}{2n^2} V_i^*.$$

Now, if it is the case that

$$U_{\text{active}}^+ - U_{\text{inactive}} = (1 - 1/n)(OPT_{+i} - OPT'_{-i}) + \frac{1}{2n^2} V_i^* - \frac{1}{n^4} OPT'_{-i} \leq 0,$$

this means that no matter what bidder i reports, being active cannot be more profitable than not being active for him. When reporting his true valuation, such a bidder will in fact be inactive, because the condition for making a bidder active is exactly $(1 - 1/n)(O - O'_{-i}) + \frac{1}{2n^2} V_i - \frac{1}{n^4} O'_{-i} > 0$, and in this case we would have $OPT_{+i} = O$ and $O'_{-i} = OPT'_{-i}$. Other than making the bidder inactive, the particular valuation he reports doesn't have an impact on his utility, so he might as well report the truth.

On the other hand, if

$$U_{\text{active}}^+ - U_{\text{inactive}} = (1 - 1/n)(OPT_{+i} - OPT'_{-i}) + \frac{1}{2n^2} V_i^* - \frac{1}{n^4} OPT'_{-i} > 0,$$

then it is more profitable for bidder i to be active, since by reporting truthfully he will get utility U_{active}^+ , better than U_{inactive} as an inactive bidder. In fact we argued above that an active bidder cannot get a better utility by reporting any valuation, so the best strategy for him is to report truthfully. In conclusion, the idealized mechanism rewards a truthfully reporting bidder by utility $\max\{U_{\text{active}}^+, U_{\text{inactive}}\}$ and that's the best the bidder can possibly receive.

Finally, we have to deal with the fact that $O = \sum_{j=1}^n O_j$ is not exactly equal to OPT , and O'_{-i} is not exactly equal to OPT'_{-i} . By Lemma 3.2, the estimates O_{+i} and O'_{-i} are in expectation within

$O(\epsilon n^7 V_i^*)$ of OPT_{+i} and OPT_{-i} . The estimates $\sum_{j \neq i} O_j$ of $\sum_{j \neq i} \mathbf{E}[v_j(S_j)]$ are strongly concentrated, let's say with high probability within $\epsilon \sum_{j \neq i} V_j = O(\epsilon n^7 V_i^*)$ of the expectation. Also, the actual social welfare of the distribution returned by the mechanism when bidder i reports truthfully is within $O(\epsilon n^7 V_i^*)$ of OPT_{+i} in expectation. Therefore, the expected utility of a truthfully reporting bidder is at least $\max\{U_{active}^+, U_{inactive}\} - O(\epsilon n^7 V_i^*)$. On the other hand, the expected utility under any other reported valuation cannot be better than $\max\{U_{active}^+, U_{inactive}\} + O(\epsilon n^7 V_i^*)$, again due to the precision of the estimates stated above. We have also ensured that the bidder's utility is at least $\frac{1}{2n^2} V_i^*$. Therefore, the relative error in utility maximization is $O(\epsilon n^9)$. \square

Now we can prove Theorem 1.6.

Proof. Using a $(1 - \epsilon)$ -MIDR mechanism \mathcal{M} , we implement a new mechanism \mathcal{M}' as above. By Lemma 3.1 and 3.3, each bidder maximizes his utility within a factor of $1 - O(\epsilon \cdot \text{poly}(n))$ by reporting truthfully. Moreover, the expected social welfare provided by mechanism \mathcal{M}' is at least $(1 - 1/n)$ times the social welfare of all *active* bidders in \mathcal{M} . (Just considering the option that we used the VCG-based allocation.) It remains to estimate the loss in social welfare due to inactive bidders.

Consider a bidder i such that $V_i \geq \frac{4}{n^2} OPT'_{-i}$. Since $OPT'_{-i} \geq \mathbf{E}[O'_{-i}]$, and O'_{-i} is a strongly concentrated estimate, with high probability we also have $V_i \geq \frac{2}{n^2} O'_{-i}$. Therefore, with high probability the bidder will be active and participate in the VCG scheme. The only bidders who do not participate with significant probability are those such that $V_i < \frac{4}{n^2} OPT'_{-i} \leq \frac{4}{n^2} OPT$. Therefore, all such bidders together cannot amount to more than $\frac{4}{n} OPT$. Overall, we recover at least a $(1 - O(1/n))$ -fraction of the social welfare achieved by mechanism \mathcal{M} , which means an approximation factor at least $(1 - O(1/n))c$. It is easy to see that the $O(1/n)$ term can be replaced by any inverse polynomial in m, n , if desired. \square

4 #P-hardness of Lottery Value Queries

Here we show that for matroid rank functions, lottery-value queries are #P-hard to answer, and require an exponential number of queries if the matroid is given by an independence oracle. We note that a lottery-value query for the vector $\mathbf{x} = (\frac{1}{2}, \dots, \frac{1}{2})$ is simply an expectation over a uniformly random set of elements.

Theorem 4.1. *There is a class of succinctly represented matroids for which it is #P-hard to compute $\mathbf{E}[r_{\mathcal{M}}(R)]$, where $r_{\mathcal{M}}$ is the rank function of \mathcal{M} and R is a uniformly random set of matroid elements. For matroids given by an independence oracle, computing $\mathbf{E}[r_{\mathcal{M}}(R)]$ requires an exponential number of queries.*

Proof. We use the class of “paving matroids” [4]: Let E be a ground set of size $2m$ partitioned into disjoint pairs e_1, e_2, \dots, e_m , and let $\mathcal{F} \subset \binom{[m]}{k}$ be any family of k -element subsets of $[m]$. Then the following is a matroid: $S \subseteq E$ is independent iff either $|S| < 2k$, or $|S| = 2k$ and S is *not* a union of pairs $\bigcup_{i \in F} e_i$ where $F \in \mathcal{F}$.

Using this construction, we can embed any #P-hard problem in a paving matroid \mathcal{M} . For example, consider the problem of counting perfect matchings. For a graph G with m edges and n vertices, we let $k = n/2$ and we define \mathcal{F} to be the family of k -edge subsets of edges that form a perfect matching. Then the matroid \mathcal{M} defined as above captures the structure of perfect matchings, since for any set of edges F the rank function $r_{\mathcal{M}}(F)$ tells us whether F is a perfect matching (which is the case if and only if $r_{\mathcal{M}}(\bigcup_{i \in F} e_i) = 2|F| - 1 = 2k - 1$). Also, the matroid is succinctly represented by the graph G , in the sense that given G we can easily decide whether a given set is independent in \mathcal{M} or not. The value of $r_{\mathcal{M}}(S)$ depends on the structure of S only if S is a union of k pairs e_i , otherwise it is $r_{\mathcal{M}}(S) = \min\{|S|, 2k\}$. Therefore, if we can compute the value of $\mathbf{E}[r_{\mathcal{M}}(R)] = 2^{-2m} \sum_{S \subseteq [2m]} r_{\mathcal{M}}(S)$, we can extract the number of perfect matchings by an elementary formula.

Similarly, for a paving matroid given by an independence oracle, the value of $\mathbf{E}[r_{\mathcal{M}}(R)]$ determines the size of the family \mathcal{F} , which could be any arbitrary family. We cannot compute this value unless we determine the size of \mathcal{F} , which requires querying all sets of the form $\bigcup_{i \in F} e_i$. This requires an exponential number of queries for independence in \mathcal{M} . \square

References

- [1] N. Alon and J.H. Spencer. The probabilistic method (2nd edition). Wiley Interscience, 2000.
- [2] S. Dughmi, T. Roughgarden and Q. Yan. From convex optimization to randomized mechanisms: towards optimal combinatorial auctions with submodular bidders. *Proc. of ACM STOC*, 149–158, 2011.
- [3] S. Dughmi. A truthful randomized mechanism for combinatorial public projects via convex optimization. *Proc. of EC*, 263–272, 2011.
- [4] J. Oxley. Matroid Theory, Cambridge University Press, 1992.
- [5] J. Vondrák. Optimal approximation for the submodular welfare problem in the value oracle model. *Proc. of ACM STOC*, 67–74, 2008.
- [6] J. Vondrák. Symmetry and approximability of submodular maximization problems. *Proc. of IEEE FOCS*, 251–270, 2009.